A Novel Quantum Approach to the Dynamics of Decision Making

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Abstract

We present a new quantum-markovian model of two-alternative forced choice (2AFC) decision-making. We treat the decision-making process as an accumulation of evidence between two competing alternatives, analogous to the drift diffusion model (DDM), in which the stimulus acts as a generative process, emitting bits of information that are treated as quantum particles. The particles are acted on by a landscape determined by the agent's experience with the task or stimulus, signal strength, and allocated cognitive control. We derive closed form expressions for success rates under both the interrogation and free response paradigms. Under the free response paradigm, we show that this model reduces to a Markov process with closed form response time (RT) distributions that take the form of inverse gaussians (IGs) with periodic noise characteristic to the task set. In the limit of long RT, the RT distributions become smooth, recovering true IG distributions analogous to the standard DDM.

Keywords: DDM; quantum cognition; markov decision-making; 2AFC

Introduction

We present a new quantum-markovian model of information accumulation for perceptual two-alternative forced choice (2AFC) decision making. Our model builds on a framework that treats each decision outcome as a square attractor potential, the width of which is determined by the relative automaticity of that response and the depth of which is determined by the amount of cognitive control allocated to it by the agent.

A prime example of 2AFC tasks is the dot motion task, in which a participant is presented with an image of moving dots, a portion of which move cohesively in one of two cardinal directions (we will use left or right) while the others move randomly (Shadlen & Newsome, 1996). The participant is tasked with identifying the direction of the cohesively moving dots (the target stimulus), a task made more or less difficult by the relative number of randomly moving dots (the distractor stimulus). In order to do so, participants are thought to exert cognitive control, in the form of attention favoring the processing of either the entire stimulus over distractors or of stimuli moving in one of the two directions, in the case of exceptions or a learned bias (e.g., movement in one direction is more frequent than the other), and thus parameterizing information processing to improve performance on the task.

The drift diffusion model (DDM) (Ratcliff, 1978) has been used to explain behavior in 2AFC tasks as a process of evidence accumulation toward one of two responses (e.g. left or right in case of the dot motion task). Although the DDM alone does not specify a mechanism for control and automaticity in such tasks, work building on the DDM has addressed the role of control in modulating its parameters (Bogacz et al., 2006). A fundamental feature of the DDM is the assumption that variability in performance is due to noise, which is typically treated as Gaussian. This is assumed to reflect influences on the decision process (or its parameters) that are uncorrelated with the signal or the process itself.

Quantum mechanical models have been proposed as an alternative for explaining the dynamics of the decision process, in which behavioral variation arises directly from a distribution of states characteristic to the system rather than unexplained exogenous influences. Justifications for this line of thinking are several, and include the inherent stochasticity of quantum systems, which may naturally capture the inherent stochasticity of neural systems without the requirement of additional variables; in cognitive models, this noise is commonly assumed to have a Gaussian form and added post hoc. In a quantum model, the form of the stochasticity of the system is determined by the parameters of the system, and is inherent to the model.

Previous quantum decision-making models treat an agent's cognitive state as a single quantum particle, repeatedly measured throughout the course of the decision-making process. Details of a DDM model of this type are given in (Busemeyer, Zheng, & Townsend, 2006). This single-particle model has the virtue of directly addressing interference and order effects, which arise from the uncertainty principle in physical systems and, in cognitive systems, reflect the fact that measuring the state of a decision by self-report has the effect of altering future measurements of that same decision. Here, we present a variant of the quantum model that differs from other quantum cognitive models in that, rather than treating the agent's mental state as a single quantum particle evolving in time, it treats the stimulus as a single generative process, emitting bits of information as quantum particles, acted on by a landscape determined flexibly by the agent's mental state. Because we are treating a process wherein the same stimulus-emitted particle is not subject to multiple measurements, our model predicts that this type of process (perceptual 2AFC with a single response to a single trial) does not exhibit the same order effects and interference properties of a single-particle process. However, it is a natural extension of single particle models that consider repeated value judgments and allows the two types of decision-making to share the quantum mechanical framework. Like the DDM, this approach assumes that the dynamics of decision-making behavior are determined by an integration process: the integration of detections of the quantum particles, serving as bits of information about the stimulus, that are measured by agents through the "filter" of their internal representations. Unlike the classic DDM, which treats perception as continuous, our particle-based model works in a discrete-time framework, which is in keeping with a range of recent work that shows that attention causes periodicity to arise in perception by inducing cortical oscillations that effectively bundle the stimulus into packets, passed between populations of neurons during a receptive window. A review of this theory and supporting evidence can be found in (Fries 2009). By modeling perceptions of the stimulus as a quantum particle, we mirror the distributed nature of internal representations of complete stimuli with competing components by diffuse and overlapping populations of neurons, as well as the fact that populations of neurons receiving the diffuse competing input will tend to, in the measurement process, collapse that input to a single representation as though they had only been shown one stimulus component (Reynolds, Chelazzi, & Desimone, 1999). In quantum mechanics, the property of particles exhibiting a distributed state before measurement and a collapsed one after measurement is known as wave-particle duality.

In our model, after determination of the properties of the quantum attractor landscape, the information accumulation process is easily reducible to a simple Markov chain. In the following analysis of our quantum Markov model, we find probabilities of success under the interrogation paradigm, closed form solutions for reaction times in the free response paradigm, and we show that, in the limit of time, the RT distributions of this model are given by inverse gaussian distributions comparable to those exhibited by the DDM. However, key aspects in both concept and behavior of our multi-particle model differ from both the standard continuous-time DDM and the single particle model, thus providing an opportunity to distinguish among these in future empirical studies.

Background

We begin by defining an infinite one dimensional attractor landscape populated by square attractors. Each attractor corresponds to the internal representation of a single stimulus component and corresponding response; that is, a set of rules encoded by the agent to bind a stimulus component to a response output. In this paper, we consider the limiting case of two alternatives, corresponding to a pair of attractors and the options in a 2AFC decision process (eg: if the cohesive motion is leftward, push the left arrow key) (Shadlen & Newsome, 1996). The agent's experience with the task, as well as the strength of the input signal (signal to noise ratio, in the dot motion task), determines the width, w_i, of the square attractor for each representation and, thus, its relative automaticity. The agent, having some information about the task they are expected to perform, then allocates cognitive control in such a way as to parameterize processing of stimulus information appropriately to allow for rapid and accurate performance. This is accomplished by the flexible allocation of control to both or individual attractors as suits the conditions of the experiment. To attend mor focally on the task-relevantstimuli (and away from distractors such as ambient noise), an agent may deepen both attractors; an agent given a somewhat reliable cue or that has noticed a frequency pattern in the stimuli may attend more closely to only one stimulus component, deepening the associated attractor. The relative allocation of control to each response determines the depth of its attractor, d_i. In an environment with two stimulus parameters, the entire landscape may then be described by a single potential, $V(w_1, w_2, d_1, d_2)$.

As noted in the introduction, there is evidence that attention operates as a "blinking spotlight", facilitated by oscillations in cortical activity that propagate information across different levels of processing in discrete packets (Fries, 2009). In this paper, we provide a formal interpretation of the blinking spotlight model of attention, treating incoming packet of perceptual information as a quantum particle, acted on by the landscape, and having its position measured to be within one attractor, the other, or neither, before a new particle is admitted to the system. This measurement is analogous to emitting one particle at a time into the landscape, placing an array of geiger counters within each attractor, and waiting for one to pick up the particle before admitting a new particle to the system.

Drawing parallels between a physical quantum system and cognitive systems must be done with care not to overstep the usefulness of the metaphor for modeling. To that end, we make several important assumptions in the treatment of incident stimulus information as particles. First, we assume that every agent participating in a task is capable of allocating control, in the form of attention, sufficiently to process relevant sensory information above chance; that is, we assume agents are able to attend the stimulus and understand how to respond to it. Quantum systems admit two types of state for particles: bound and scattering. It is only in bound states that particles can be expected to be found within the attractors with probability above chance. For this reason, we will treat only the bound states of our system. Second, while it is reasonable to think that the agent's state of arousal, (with respect to its influence on the focus vs. dissipation of attention) which may change between or even within trials, will impact their ability to perform the task, we cannot, within the confines of this model, make the assumption that any one state of arousal is more likely than another. In a quantum system, we may relate the agent's arousal to the specific energy of a bound state. For example, at higher energy, the particle is more likely to be found outside of an attractor, much as an agent that is more agitated is less likely to be able to attend a demanding task. Furthermore, and on a more technical note, the measured state of an agent is described by their performance metrics (RT and correctness), unlike a physical quantum system, the measured state of which often belies its energy. For these reasons, given that our model system is comprised of a single particle of unknown energy, we rely on the postulate of equal a priori probabilities to determine that each bound state admitted by the system is equally likely to occur. Finally, although allocated control is known to fluctuate both within and between trials, we assume that the fluctuations happen in between the measurements of particles (processing of a single packet of stimulus information) such that the corresponding changes in the depths of attractors do not alter the wave function of a particle between when it is admitted to the system and when it is measured; that is, the stimulus packet is passed through the system while it is in a single synchronous state, not while the synchronicity of relevant populations is in flux. This seems to be the case, not merely a simplifying assumption, for cognitive systems experiencing the synchronous effects of attention (Fries, 2009). This allows us to find the bound states and corresponding probability distributions for a particle in V by solving the time independent Schroedinger equation.

Next, we will briefly describe the form of the bound states, which gives rise to the probability distributions that determine where, in the landscape, a particle is found, and lend insight to one of the phenomena of this model, the "dropped bit", in which an incident stimulus particle is not passed along for higher processing because it has not been found to be in either attractor. Prior to measurement, the state of each piece of information, determined by the bound states, is a continuous distribution across both attractors. Although the states vary with attractor parameters, they all share a canonical form: exponentials that converge to zero as their position goes to positive or negative infinity and oscillations within the wells. The quantum system always has the property, therefore, of having a non-zero probability of the particle being found outside the attractors. This probability is higher in narrower attractors, associated with a less salient stimulus or less automatic response and in shallower attractors, to which less control has been allocated. The presence of multiple attractors of finite depths affects the distributions, causing states to have components that reside in both attractors and giving rise to a fundamental conflict between stimulus components, altering the properties of success rate and RT distributions.

In order to accumulate information about the stimulus, the agent makes measurements of the position of the quantum particle, causing the collapse of its state from a diffuse distribution to a definite state. This dynamic is similar to one found in neural populations in the visual system when distributed populations of neurons, representing a a number of stimuli in an overlapping receptive field, compete for representation by a population at a higher level of processing. It has been shown that the neurons at the higher level of processing will respond to the mixed stimulus representations as they would to only a single representation (Reynolds, Chelazzi, and Desimone, 1999). Similarly, the measurement of the quantum particle in our model allows for finding the particle to be in well 1 (associated with alternative 1), well 2 (associated with alternative 2), or neither well. This final case may be referred to as "dropping a bit", and occurs when the system does not integrate a bit in a useful way. It may be attributed to temporary inattentiveness, but always has a non-zero probability of occurrence, regardless of how much control an agent may attempt to exert. The information is then integrated, according to the attractor in which it was measured, as evidence for either choosing response 1 (leftward motion, left arrow key), choosing response 2 (rightward motion, right arrow key), or is ignored. The probability of each case is determined by integrating the L2 norm of the bound states across the width of the wells.

To understand this more intuitively, consider the dot motion stimulus. Each incoming bit contains information about multiple components of the stimuli, i.e. the direction of both the target (cohesive) stimulus and the distractor (random) stimulus.The agent processes the components of the stimulus at each moment according to how much experience they have perceiving and responding to such stimuli (the width of the wells), their commitment to performing the task (their ability to deepen both attractors) as well as how much control they have allocated to favoring the processing of one stimulus over the other to perform the task at hand (their tendency do deepen a singular attractor if given a reason, such as asymmetric stimulus frequency, to do so). This combination of automaticity and expressions of control creates a two well "filter" that shapes the probability distributions for the stimuli. Then, by making judgments about the stimulus, the agent collapses those distributions, and integrates the information, stochastically, as perception of left- or right-moving dots. The measurement clears the particle from the system by passing the bit to a higher level of processing, similarly to a geiger counter clearing a photon in measurement by converting its energy to current. Repeated measurements of the stimulus lead to the accumulation of evidence and, when one stimulus component has sufficiently out-paced the other, the agent responds.

Theory and Results

Having addressed the qualitative aspects of this DDM model, we now continue to a rigorous analysis of its mathematical properties, yielding closed forms for the probability of success under the interrogation paradigm as well as the probability of success and the RT distributions under the free response paradigm. These results mirror the DDM in limiting cases, but have certain characteristics that arise directly from the theoretical framework employed here.

Relating the Multi-Particle Quantum DDM to the Traditional DDM

In the standard DDM, the drift, dx is given by Brownian motion, where the displacement of the decision variable with respect to its starting point is the sum of a series of independent, identically distributed (IID) Gaussian variables. The resultant instantaneous displacement is given by

$$dx = Adt + cdW \tag{1}$$

Where A is the mean drift rate and the second term is normally distributed noise (Bogacz et al., 2006).

In our model, the displacement of the decision variable is also the sum of IID variables, whose probability distributions are given by the probability of a bit being accumulated as evidence for either response 1, response 2, or neither.

Value assignments for each bit take on the value of a step size for a random walk

$$dx = x_t \in \{-1, 0, 1\} \tag{2}$$

And the agent's progress from their initial position at time is given by

$$X(\tau) = \sum_{t=1}^{\tau} x_t \tag{3}$$

The probabiliy of each value of x_t is drawn from a uniformly distributed set of probabilities, which arise from a set of equally probable bound eigenstates admitted by V. We denote the probability that step x_t has a value a, given the eigenstate j as,

$$\mathbb{P}_{aj} = \mathbb{P}(x_t = a|j) \tag{4}$$

, where $j \in [1, 2, 3, ...n]$ is the index of the eigenstate.

The expectation of x_t, analogous to the mean drift rate of the standard DDM, is therefore given by

$$M = \frac{1}{n}\left(\sum_j^n \mathbb{P}_{1j}\right) - \frac{1}{n}\left(\sum_j^n \mathbb{P}_{-1j}\right) = M_1 - M_2 \in [-1, 1] \tag{5}$$

The stochastic trajectory of this model arises directly from the competition between target and distractor stimulus-associated attractors in the landscape, each of which exerts an attractive force on the particle, laying claim to some part of the probability distribution.

State of the System Over Time

This double-well quantum variant on the DDM evolves as a discrete time Markov chain with a (possibly infinite) single-step transition matrix, which gives the probability of transitioning from state i to state j in a single step, given by

$$T = \begin{bmatrix} 1-(M_2+M_1) & M_2 & 0 & 0 & \ldots & 0 \\ M_1 & 1-(M_2+M_1) & M_2 & 0 & \ldots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & M_1 & 1-(M_2+M_1) \end{bmatrix} \tag{6}$$

The two common approaches to 2AFC tasks are the interrogation paradigm and the free response paradigm. Under the interrogation paradigm, the agent is exposed to the stimulus until a time determined by the experimenter, and then forced to respond. Under the free response paradigm, the agent is instructed to respond as quickly and accurately as possible, then exposed to the stimulus until they decide to respond, presumably when the system hits one of two decision boundaries. Under the interrogation paradigm, we are concerned only with probability of a correct choice. Under free response, we are interested in both probability of a correct choice and response time, as well as the tradeoff between speed and accuracy. For the double-well quantum DDM, the interrogation and free response paradigms call for different analyses.

Interrogation Paradigm

Under the interrogation paradigm, we allow the system to evolve freely and take a measurement of its state at the time determined by the experimenter, t. As with the standard DDM, the probability of success given by

$$\mathbb{P}_S = \mathbb{P}(X > 0) \tag{7}$$

In order to find these values, we need to know the probability that the system is in a state X_i given that it started in state X_0.

Since the system's step size is bounded above and below, the state of the system is also bounded above and below.

$$x_t \in \{-1, 0, 1\} \tag{8}$$

$$X \in \{-t,\ -t+1,\ \ldots\ 0,\ \ldots\ t-1,\ t\} \tag{9}$$

We find the probability distribution of the state of the system across all possible states by fixing the transition matrix such that $T \in \mathbb{R}^{n,n}$

$$n = 2t + 1 \tag{10}$$

And finding

$$S_t = T^t X_0 \tag{11}$$

$$X_0 = \hat{e}_{t+1} \tag{12}$$

Where $\hat{e}_{t+1} \in \mathbb{R}^n$ is the characteristic vector, with all entries zero except the $t+1$st entry, which is 1, and corresponds to starting the system at time $t = 0$. S_t is the probability distribution for all possible states of the system, X.

T is a finite Toeplitz tridiagonal matrix with constants on the diagonals, which allows for analysis of arbitrary positive integer powers (Salkuyeh, 2006). In our special case, the (i,j)th entries of T are given by,

$$T^t_{ij} = \left(\frac{1}{t+1}\right)\left(\frac{M_1}{M_2}\right)^{\frac{i-j}{2}} \sum_{k=1}^{2t+1} \lambda_k^t sin\left(\frac{i\pi k}{2(t+1)}\right) sin\left(\frac{j\pi k}{2(t+1)}\right) \tag{13}$$

$$\lambda_k = (1 - M_1 - M_2) + 2\sqrt{M_1 M_2} cos\left(\frac{k\pi}{2(t+1)}\right) \tag{14}$$

And, since the system began at $t = 0$, its probability distribution at time t is described by the entries of the $t+1$th column of T^t

$$T^t_{i,t+1} = \left(\frac{1}{t+1}\right)\left(\frac{M_1}{M_2}\right)^{\frac{i-t-1}{2}} \sum_{a=0}^{t} \lambda^t_{2a+1} sin\left(\frac{\pi i(2a+1)}{2(t+1)}\right)(-1)^a \tag{15}$$

This is a gaussian-like distribution with periodic interference. As time progresses, the interference dissipates, and we recover a gaussian distribution.

Assigning the positive direction to be the correct one, the probability of success is given by

$$\mathbb{P}_S = \sum_{i=t+1}^{2t+1} \left(T^t_{i,t+1} \right) \tag{16}$$

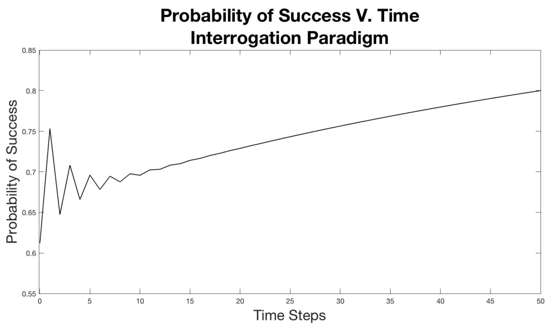

Figure 1: A sample of the probabilities of success over time under the interrogation paradigm, $M_1 = 0.488$, $M_2 = 0.388$. As expected, it will converge to 1 in the limit of time, but exhibits rapid oscillations and a slower non-monotonicity that depend on M_1 and M_2. In the case of high conflict, there is a chance for the outcomes to cross the axis of equal probability due to these oscillations.

Free Response Paradigm

Under the free response paradigm, we set two decision boundaries at $x = \{-T_f, T_s\}$ where $-T_f$ is the threshold for making the incorrect decision (failure), and T_s is the threshold for making the correct decision (success). We do not assume that these boundaries are symmetric, which corresponds to allowing for the existence of bias in a standard DDM. The system now evolves as a Markov chain with two absorbing states. Its one-step transition probability matrix is given by

$$A = \begin{bmatrix} 1 & M_2 & 0 & \ldots & 0 & 0 \\ 0 & 1-(M_2+M_1) & M_2 & 0 & 0 & \ldots \\ 0 & M_1 & 1-(M_2+M_1) & M_2 & 0 & \ldots \\ \vdots & & & & & \\ 0 & 0 & 0 & M_1 & 1-(M_2+M_1) & 0 \\ 0 & 0 & 0 & 0 & M_1 & 1 \end{bmatrix} \in \mathbb{R}^{mxm} \tag{17}$$

where

$$m = T_f + T_s + 1 \tag{18}$$

The system may occupy any of state of a discrete space given by $x \in R^{mx1}$. The state of the system at time t is again given by

$$S_t = A^t S_0 \tag{19}$$

. The probabilities of success and failure can be described by setting the thresholds for each choice as absorbing states and finding the probability of absorption by each from the starting point.

The probabilities of success and failure are found by solving the expressions

$$Bx_f = f \tag{20}$$

.

$$Bx_s = s \tag{21}$$

. Where $s = \hat{e}_m$ and $f = \hat{e}_1$ for x_f and x_s. The entries $(x_s)_i$ give the probability of success, having started from $x_i \in [-T_f, -T_f+1, \ldots T_s - 1, T_s]$, and failure is given likewise by $(x_f)_i$. The matrix B is

$$B = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ M_2 & -(M_2+M_1) & M_1 & 0 & 0 & \ldots \\ 0 & M_2 & -(M_2+M_1) & M_1 & 0 & \ldots \\ \vdots & & & & & \\ 0 & 0 & 0 & M_2 & -(M_2+M_1) & M_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

And the boundary conditions are given by $(x_s)_1 = 0$, $(x_s)_m = 1$ and $(x_f)_1 = 1$, $(x_f)_m = 0$. That is, if the agent begins exactly at the threshold for the "incorrect" decision, their probability of success is zero and their probability of failure is 1.

The entries of x_s and x_f are exponential and given by,

$$(x_f)_i = C_{1f} + C_{2f} \left(\frac{M_2}{M_1} \right)^i \tag{23}$$

$$C_{1f} = \frac{-M_2^{m-1}}{M_1^{m-1} - M_2^{m-1}} \tag{24}$$

$$C_{2f} = \left(\frac{M_1}{M_2} \right) \left(\frac{M_1^{m-1}}{M_1^{m-1} - M_2^{m-1}} \right) \tag{25}$$

And

$$(x_s)_i = C_{1s} + C_{2s} \left(\frac{M_2}{M_1} \right)^i \tag{26}$$

$$C_{1s} = \frac{M_1^{m-1}}{M_1^{m-1} - M_2^{m-1}} \tag{27}$$

$$C_{2s} = \left(\frac{-M_1}{M_2} \right) \left(\frac{M_1^{m-1}}{M_1^{m-1} - M_2^{m-1}} \right) \tag{28}$$

With these values in hand, we can now set the initial position of the agent to $x = 0$, and the probability of success and failure are given by the T_{f+1}th entries of $x_f$and x_s.

$$\mathbb{P}_S = M_1^{T_s} \left(\frac{M_1^{T_f} - M_2^{T_f}}{M_1^{T_f+T_s} - M_2^{T_f+T_s}} \right) \tag{29}$$

The probability of the agent choosing incorrectly is defined and found similarly.

$$\mathbb{P}_F = M_2^{T_f} \left(\frac{M_1^{T_s} - M_2^{T_s}}{M_1^{T_f+T_s} - M_2^{T_f+T_s}} \right) \tag{30}$$

Where $M_1 \neq M_2$. Where $M_1 = M_2$, $\mathbb{P}_F = \mathbb{P}_S = 0.5$. Success rates follow a binomial distribution.

Mean Response Time A useful parameter for describing an agent's performance is the mean response time for both correct and incorrect responses, analogous to the unconditioned RT in the standard DDM.

This is done by setting absorbing states at the boundaries $-T_f$ and T_s . Next we find the characteristic matrix $N = (I - Q)^{-1}$ where $Q \in R^{m-2xm-2}$ s.t.

$$Q = \begin{bmatrix} 1-(M_1+M_2) & M_1 & 0 & \dots 0 \\ M_2 & 1-(M_1+M_2) & M_1 & \dots 0 \\ 0 & M_2 & 1-(M_1+M_2) & \dots 0 \\ \vdots & & & \end{bmatrix} \tag{31}$$

$$N_{ij} = \begin{cases} -1^{i+j}\left(\dfrac{-M_1^{j-i}}{\sqrt{M_1M_2}^{j-i+1}}\right)\left(\dfrac{U_{i-1}(d)U_{m-j}(d)}{U_m(d)}\right) & i \leq j \\ -1^{i+j}\left(\dfrac{-M_2^{i-j}}{\sqrt{M_1M_2}^{i-j+1}}\right)\left(\dfrac{U_{j-1}(d)U_{m-i}(d)}{U_m(d)}\right) & j < i \end{cases} \tag{32}$$

Where $U_i(d)$ is the ith Chebyshev polynomial of the second kind, evaluated at d and d is given by

$$d = \frac{M_1 + M_2}{2\sqrt{M_1M_2}} \tag{33}$$

(Fonesca & Petronilho, 2005). The sum over the entries of each row of N gives the expected time before absorption by either state, having started in state i . Since our agent always begins at $t = 0$, we are only interested in $i = T_f + 1$. Therefore, the mean response time for the system is given by

$$\mathbb{E}(RT) = \sum_{j=1}^{m-2} N_{T_f+1,j} \tag{34}$$

.

RT Distributions as a First Passage Problem To get the RT distributions for the free response paradigm, we must consider a first passage problem. The probabilities that the first passage time from initial state S_0 to final states s and f is t will be denoted by $f_s(t)$ and $f_f(t)$ respectively.

To find $f_s(t)$, we begin by setting our decision thresholds as absorbing states in our Markov chain. Thus, the single-step transition probabilities are given by A (as defined above). Probability of absorption by success and failure thresholds respectively at time is given by

$$a_s(t) = (A^t S_0)_m = (S_t)_m \tag{35}$$

And

$$a_f(t) = (A^t S_0)_1 = (S_t)_1 \tag{36}$$

. However, $a_s(t)$ and $a_f(t)$ give the probabilities of having been absorbed at time t and all preceding times. Therefore,

$$f_s(t) = a_s(t) - a_s(t-1) \tag{37}$$

$$f_f(t) = a_f(t) - a_f(t-1) \tag{38}$$

The tridiagonal nature of A allows closed-forms of the distributions $f_s(t)$ and $f_f(t)$. To find these, we reduce to a matrix of matrices,

$$A = \begin{bmatrix} 1 & B & 0 \\ 0 & T & 0 \\ 0 & F & 1 \end{bmatrix} \in \mathbb{R}^{mxm} \tag{39}$$

$$B = \begin{bmatrix} M_2 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{1\times m-2} \tag{40}$$

$$F = \begin{bmatrix} 0 & 0 & \dots & M_1 \end{bmatrix} \in \mathbb{R}^{1\times m-2} \tag{41}$$

And $T \in \mathbb{R}^{m-2\times m-2}$ is our familiar Toeplitz matrix.

$$A^t - A^{t-1} = \begin{bmatrix} 0 & BT^{t-1} & 0 \\ 0 & T^t - T^{t-1} & 0 \\ 0 & FT^{t-1} & 0 \end{bmatrix} \tag{42}$$

Because we are only interested in probabilities of being in either absorbing state, we are interested only in the top and bottom rows of this matrix, the $T_F + 1$st entries of which give the probability of succeeding or failing at time t.

$$f_s(t) = M_1(T^{t-1})_{(m-2,T_f)} \tag{43}$$

$$f_f(t) = M_2(T^{t-1})_{(1,T_f)} \tag{44}$$

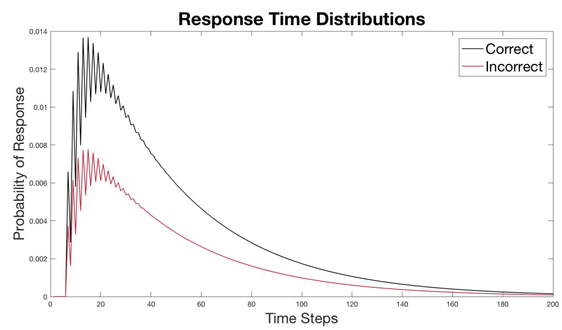

Figure 2: A sample of the RT distributions for success and failure, $M_1 = 0.488$, $M_2 = 0.45$. These distributions oscillate inside of inverse gaussian envelopes. The periodicity they exhibit is due to the asymmetry of M_1 and M_2 and the discrete measure of time.

These distributions are shaped like inverse gaussian distributions, but exhibit periodic noise characteristic to the system.

RT Distributions in the Limit of Time Because the state of the system is a sum of IID random variables, we know by the central limit theorem that the success rate distributions will converge to a Gaussian

$$P(S) = \mathcal{N}\left(M, \frac{M_1 + M_2 - M^2}{n}\right) \tag{45}$$

And that the probability distribution of the states for the interrogation paradigm (no absorbing states) as the interrogation time increases is also bounded by a Gaussian distribution such that

$$\lim_{n\to\inf} S_n = \mathcal{N}\left(Mn, n\left(M_1 + M_2 - M^2\right)\right) \tag{46}$$

Thus, in the limit of sufficiently high thresholds or sufficiently long time, the RT distributions are given by an inverse gaussian distribution for the limiting state S_n,

$$RT = IG\left(\frac{T_S}{M}, \frac{T_S^2}{M_1 + M_2 - M^2}\right) \tag{47}$$

Conclusion

We have presented a novel discrete time quantum-markov model for 2AFC decision-making processes that yields closed form success and failure probabilities and response time distributions. The model's discrete treatment of time arises directly from the treatment of incident stimuli as quantum particles of information, which are propagated in a periodic way and begins to provide a formally rigorous account of the "blinking spotlight" effect that attention lends to perception (Fries, 2009). The motivation for the quantum nature of the particles is justified both by previous models that yield quantum-like results for judgment decision processes, as reviewed in the work of (Busemeyer, Wang, & Pothos) and by the dynamics of neural representations as information passes through different levels of processing, which seems to mirror the wave-particle duality of quantum particles in that distributed representations are collapsed to distinct ones by the measurement process (Reynolds, Chelazzi, & Desimone, 1999).

The RT distributions are analogous in form to the inverse gaussians familiar from a standard DDM (Bogacz et al., 2006), but differ in that the noise that is inherent to the system and determined by the agent's task set and experience, as well as the stimulus itself. The noise is intrinsic to the system in that it is determined by M_1 and M_2, which are determined by the quantum attractor landscape V, rather than being generalized to being Gaussian. In the limit of time or sufficiently high decision thresholds, the noise of the system dissipates and we recover inverse gaussian distributions but, on shorter time scales, the system exhibits oscillatory components, arising as a result of theoretical differences from the standard DDM, that make novel predictions that may be measurable by future experiments. Because of their closed form, RT distributions and success probabilities are quick and computationally cheap to calculate, as they do not require simulation.

This model is a limiting case arising from a framework that treats incoming stimulus information as quantum particles subject to attractor potentials determined by the agent as well as the stimulus. In future work, this framework will be extended to include dynamics of control allocation and will be used to treat task-switching applications and higher dimensional stimuli. Current work is extending this framework for analysis of multi-choice decision making processes and 2AFC with bivalent stimuli, which requires greater levels of processing.

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